

# MCD-FINITE DOMAINS AND ASCENT OF IDF PROPERTY IN POLYNOMIAL EXTENSIONS

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**ABSTRACT.** An integral domain is said to have the IDF property, when every non-zero element of it, has only a finite number of non-associate irreducible divisors. It was first Malcolmson and Okoh, who found a counterexample showing that IDF property does not necessarily ascend in polynomial extension. In this paper, we introduce a new class of integral domains, called MCD-finite domains and show that for any domain  $D$ ,  $D[X]$  is an IDF domain if and only if  $D$  is both IDF and MCD-finite. Then, we use this theorem together with a construction introduced by Roitman to find more counterexamples to the ascent of IDF property in polynomial extension.

## 1. INTRODUCTION

An integral domain is called IDF, if every non-zero element of it, has only a finite number of non-associate irreducible divisors. These domains were introduced by Grams and Warner in [12]. They are also one of the generalizations of UFD's that were studied in the seminal paper [1]. Another important subclass of IDF domains, are domains that contain no atoms at all; these domains were named antimatter and studied in [8]. A question first posed in [1], is that whether IDF property ascends in polynomial extension. Malcolmson and Okoh in [13], answered to this question in the negative. They actually proved that:

**Theorem 1.1** ([13, Theorem 2.5]). *Every countable domain can be embedded in a countable antimatter domain  $R_\infty$ , such that  $R_\infty[X]$  is not an IDF domain.*

A natural question that follows, is that under which additional conditions, IDF property does ascend in polynomial extensions. One such condition is when the domain, in addition to being IDF, is a valuation domain or more generally a GCD domain ([1, p 14] and [13, Theorem 1.9]). Another case is when the domain is atomic. In fact, atomic IDF domains are exactly FFD's ([1, Theorem 5.1]) and this class of domains ascend in polynomial extension ([1, Proposition 5.3]). Anderson, Anderson and Zafrullah in [1], showed that if  $D[X]$  is an atomic domain, then any two non-zero elements in  $D$ , must have a maximal common divisor (MCD);

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i.e,  $D$  must be a weak GCD domain ([1, Theorem 1.3]; For stronger results, see [1, Theorem 1.4] and [15, Proposition 1.1]). This fact was used by Roitman in [15] to find an example of an atomic domain, such that its polynomial extension is not atomic ([15, Example 5.1]).

Due to the importance of the construction introduced by Roitman, here we sketch an outline of it and some of its applications. We must point out that in this paragraph, we have deliberately eschewed the technicalities and pitfalls buried in this construction, so that we could give the big picture. The interested reader may refer to [3], [9], [10] and [15] for the details. Let  $D$  be a domain,  $S$  the set of all reducible elements of  $D$  and  $R = D[\{X_s, \frac{s}{X_s} \mid s \in S\}]$ . Then every element  $s$  of  $S$ , has an atomic factorization in  $R$ ; that is  $s = (X_s)(s/X_s)$ . Applying this extension inductively and by taking the union of the resulting chain, he constructed an atomic domain  $\mathcal{A}^\infty$ . The important property of the extension  $D \subseteq \mathcal{A}^\infty$ , is that the MCD's of any set of non-zero elements of  $D$ , is preserved in  $\mathcal{A}^\infty$  and hence, if  $D$  is not an MCD domain, then neither is  $\mathcal{A}^\infty$  (See [15, Section 3]). So, if we apply this extension to a domain which is not weak GCD domain, the resulting domain would be a counterexample to the ascend of atomic property, and this is what Roitman did in [15, Example 5.1]. If we let  $S$  to be a subset of atoms of  $D$ , then this technique can also be used to “discard” an arbitrary set of atoms of  $D$  by turning them into reducible elements. This method was first used by Coykendall and Zafrullah in [10] to construct an U-UFD domain that is not AP. In fact, they (inductively) chose  $S$  to be the set of all the irreducible elements of  $D$ , except one atom  $\pi$  that is not a prime. The resulting domain contains exactly one non-prime atom (i.e.,  $\pi$ ) and hence is a U-UFD that is not AP ([10, Theorem 2.8]). This technique also was used by Coykendall and Mammenga in [9] to show that, for every cancellative, torsion-free, reduced atomic monoid  $S$ , there exists a domain with the same “atomic factorization structure” as  $S$  (But which is not necessarily atomic). This was achieved by discarding all the atoms of the monoid domain  $R[X; S]$ , except non-constant monomial ones, where  $R$  is an arbitrary domain ([9, Theorem 3.3]). Most recently, using this construction, Rand in [14] proved that for any domain  $D$  and any unit-closed subset  $S$  of atoms of  $D$ , there exists a domain  $R$  containing  $D$ , such that the set of atoms of  $R$  is exactly  $S$  ([14, Theorem 2.7] and Theorem 4.1).

In this paper, we introduce the class of MCD-finite domains (Domains in which, any finite set of non-zero elements, has only a finite number of non-associate MCD's) and in Theorem 2.1, we show that for any domain  $D$ ,  $D[X]$  is IDF domain if and only if  $D$  is IDF and MCD-finite. In Section 2, we provide some examples regarding MCD and MCD-finite domains. Specially, we are interested in examples of domains that are not MCD-finite, and in fact, we present methods by which it is possible to embed a domain  $D$  in a domain  $R$  that is not MCD-finite (Theorem 3.8, Proposition 3.12 and Example 3.14). Finally in section 3, we use Rand's theorem to present a stronger version of Theorem 1.1 (Theorem 4.2). In the remainder of this section, we briefly state some preliminaries that are needed for the rest of the paper.

For a domain  $D$ , the set of its non-zero elements, units, non-zero non-units and field of fraction will be denoted by  $D^*$ ,  $U(D)$ ,  $D^\#$  and  $\text{frac}(D)$  respectively. Also following the notation and terminology of [14], the set of atoms of  $D$  will be denoted by  $\mathcal{A}_D$  and we call a subset  $A$  of  $D$  unit-closed, if for every  $u \in U(D)$  and  $a \in A$ , we have  $ua \in A$ . Also for  $a, b \in D$ , we use the notations  $a \sim_D b$  and  $a \mid_D b$  to emphasize the domain in which this relations hold.

A domain  $D$  is called atomic, if every  $x \in D^\#$  can be written as a product of irreducible elements (atoms). If the set of principal ideals of  $D$  satisfies the ascending chain condition, then  $D$  is called an ACCP domain. Finally, if for every  $x \in D^*$ , the number of divisors of  $x$ , up to associates, is finite, then  $D$  is called a finite factorization domain (FFD). For more on these and other domains with factorization properties, see [1].

Let  $D$  be a domain and  $S \subseteq D^*$ . The set of all the common divisors of  $S$ , is denoted by  $\text{CD}_D(S)$ . An element  $c$  of  $\text{CD}_D(S)$ , is called a maximal common divisor of  $S$ , whenever for every  $d \in \text{CD}_D(S)$ , if  $c \mid d$ , then  $c \sim d$ . The set of all the maximal common divisors of  $S$  is denoted by  $\text{MCD}_D(S)$ . If every finite subset of  $D^*$  has an MCD, then  $D$  is called an MCD domain. Also, we call  $D$  MCD-finite, if the number of MCD's of every non-associate finite subset of  $D$  is finite (and possibly empty). We say  $a, b \in D^*$  are incomparable if  $a \nmid b$  and  $b \nmid a$  (i.e, the principal ideals  $\langle a \rangle$  and  $\langle b \rangle$  are incomparable).

A commutative (additive) monoid  $T$  is called cancellative, if for every  $a, b, c \in T$ , if  $a + b = a + c$ , then  $b = c$ . Also  $T$  is torsion-free, if for every  $a, b \in T$  and  $n \in \mathbb{N}$ , if  $na = nb$ , then  $a = b$ . Finally,  $T$  is reduced if it has no non-trivial units; i.e., if  $a + b = 0$  for some  $a, b \in S$ , then  $a = b = 0$ . All the concepts defined in the two previous paragraphs, can be extended to cancellative monoids in an obvious way. We also recall that the monoid ring  $R[X; T]$  is a domain if and only if  $T$  is cancellative and torsion-free and  $D$  is a domain [11, Theorem 8.1].

We say that an extension of domains  $A \subseteq B$ , is division-preserving, if for every  $x, y \in A^*$ , if  $x \mid_B y$ , then  $x \mid_A y$ . This can also be stated concisely by saying  $B \cap \text{frac}(A) = A$  (For some properties of these extensions that are relevant to this paper, see [15, Remark 2.2]).

Let  $R$  be a ring and  $f = \sum_{i \in \mathbb{N} \cup \{0\}} a_i X^i \in R[X]$ . The leading coefficient of  $f$  is denoted by  $\text{lc}(f)$ , and the support of  $f$  is defined as the set  $\{X^i \mid i \in \mathbb{N} \cup \{0\} \text{ and } a_i \neq 0\}$  and is denoted by  $\text{supp}(f)$ .

## 2. MAIN RESULT

We begin with the following theorem, which states a necessary and sufficient condition for the ascent of IDF property in polynomial extensions.

**Theorem 2.1.** *The following are equivalent for a domain  $D$ :*

- (1)  *$D$  is IDF and MCD-finite.*
- (2)  *$D[X]$  is IDF.*
- (3) *For any set of variables  $\{X_a\}_{a \in \alpha}$ ,  $D[\{X_a\}_{a \in \alpha}]$  is IDF.*

*Proof.* (1  $\implies$  2) Let  $D$  be an IDF domain such that  $D[X]$  is not IDF. We will show that  $D$  is not MCD-finite. Let  $f \in D[X]^\#$  be such that there exists an infinite set  $\{f_i\}_{i \in I}$  of non-associate irreducible divisors of  $f$ .

Let  $K$  be the field of fractions of  $D$ . Since  $K[X]$  is a UFD, there exists an infinite subset  $J$  of  $I$ , such that the elements of  $\{f_i\}_{i \in J}$  are associate in  $K[X]$ . So in particular, the supports of  $f_i$ 's for every  $i \in J$  are the same. Also, note that since  $D$  is IDF, we can assume that for every  $i \in J$ ,  $2 \leq |\text{supp}(f_i)|$  and hence, the GCD of the coefficients of  $f_i$  is equal to 1.

For every pair of unequal elements  $i$  and  $j$  in  $J$ , there exists an element  $u_{i,j} \in K^*$ , such that  $f_i u_{i,j} = f_j$ , and so  $u_{i,j} = \frac{\text{lc}(f_j)}{\text{lc}(f_i)}$ . Now fix an element  $t$  of  $J$ . Then, for

every  $i \in J$  such that  $i \neq t$ , we have  $f_t = \frac{\text{lc}(f_i)}{\text{lc}(f_i)} f_i$ , and so  $\text{lc}(f)f_t = \text{lc}(f)\frac{\text{lc}(f_i)}{\text{lc}(f_i)} f_i$ . Since  $\text{lc}(f_i) \mid_D \text{lc}(f)$ , we have  $\text{lc}(f)\frac{\text{lc}(f_i)}{\text{lc}(f_i)} \in D^*$ . Also if  $i \neq j$ , then  $\text{lc}(f)\frac{\text{lc}(f_i)}{\text{lc}(f_i)} \not\sim_D \text{lc}(f)\frac{\text{lc}(f_j)}{\text{lc}(f_j)}$ ; otherwise  $\text{lc}(f_i) \sim_D \text{lc}(f_j)$ , so  $u_{i,j} \in U(D)$  and so  $f_i \sim_{D[X]} f_j$ , which is a contradiction. Since for every  $i \in J$ , the GCD of the coefficients of  $f_i$  is equal to 1, it follows that  $\{\text{lc}(f)\frac{\text{lc}(f_i)}{\text{lc}(f_i)}\}_{i \in J}$  is an infinite set of non-associate maximal common divisors of the coefficients of  $\text{lc}(f)f_t$ , and hence  $D$  is not MCD-finite.

(2  $\implies$  1) If  $D$  is not IDF, then obviously neither is  $D[X]$ .

Now let both of the domains  $D$  and  $D[X]$  be IDF domains, but there exists  $n \in \mathbb{N}$ , and elements  $a_0, a_1, \dots, a_n$  in  $D^*$ , with infinitely many non-associate maximal common divisors  $\{c_i\}_{i \in I}$ .

Let  $f = a_0 + a_1X + \dots + a_nX^n$ . For every  $i \in I$ , there exists an  $f_i \in D[X]^\#$ , such that  $f = c_i f_i$ . Because the GCD of the coefficient of each  $f_i$  is equal to 1, each  $f_i$  has an atomic factorization in  $D[X]$ . But every irreducible divisor of each of the  $f_i$ 's is an irreducible divisor of  $f$  too. Therefore, since  $D[X]$  is an IDF domain, there exists a finite set of non-associate irreducible divisors  $\{g_1, \dots, g_m\}$  of  $f$  such that, every  $f_i$ , up to associates, is equal to a product of  $g_i$ 's. So for every  $i \in I$ , there exists  $t_{1,i}, \dots, t_{m,i} \in \mathbb{N} \cup \{0\}$ , such that  $f_i \sim_{D[X]} g_1^{t_{1,i}} \dots g_m^{t_{m,i}}$ . Note that for every  $1 \leq j \leq m$ , we have  $1 \leq \deg(g_j)$ , because the GCD of coefficients of  $f_i$  is 1. So, for every  $1 \leq j \leq m$ , the set  $\{t_{j,i}\}_{i \in I}$  is finite and so there exist  $k, \ell \in \mathbb{N}$  such that  $k \neq \ell$  and  $g_1^{t_{1,k}} \dots g_m^{t_{m,k}} = g_1^{t_{1,\ell}} \dots g_m^{t_{m,\ell}}$ , and so  $f_k \sim_{D[X]} f_\ell$ . Hence  $c_k \sim_{D[X]} c_\ell$  and so  $c_k \sim_D c_\ell$  which is a contradiction.

The proof of (3  $\implies$  1) and (1  $\implies$  3), are similar to implications which have been proved. □

It is worthwhile to compare the notions of IDF and MCD-finite domains, ideal-theoretically.

**Proposition 2.2.** (1) ([1, p.12]) *A domain  $D$  is IDF if and only if for any non-zero (finitely generated) ideal  $I$ , there only exists a finite number of ideals containing  $I$  that are maximal with respect to being principal.*  
 (2) *A domain  $D$  is MCD-finite if and only if for any finitely generated ideal  $I$ , there only exists a finite number of ideals containing  $I$  that are minimal with respect to being principal.*

### 3. EXAMPLES

In this section, we provide some examples and counterexamples regarding MCD and MCD-finite domains. We begin with mentioning an important subclass of MCD domains.

**Proposition 3.1.** *Any ACCP domain is an MCD domain.*

*Proof.* If  $D$  is an ACCP Domain, then  $D[X, Y]$  is also an ACCP domain and hence is atomic and so by [1, Theorem 1.4]  $D$  is an MCD Domain.

This fact can also be proven directly. Suppose  $D$  is not an MCD domain. Then there exist  $c_1, \dots, c_n \in D^\#$ , such that  $\text{MCD}(c_1, \dots, c_n) = \emptyset$ . But then, there exists an infinite descending chain of principal ideals  $b_1D \supsetneq b_2D \supsetneq \dots$  all of which contain  $c_i$ 's. But if  $D$  is an ACCP domain, then for any strictly descending chain

of principal ideals  $a_1D \supsetneq a_2D \supsetneq \cdots$  we have  $\bigcap_{i \in \mathbb{N}} a_iD = 0$  ([1, p.4]) which is a contradiction.  $\square$

A trivial example of MCD-finite domains are FFD's. In particular, every Krull domain is an MCD-finite domain ([1, p.14]). In fact, the stronger result in [12], can be extended to the case of MCD-finite domains as follows:

**Theorem 3.2** ([12, Proposition 1]). *Let  $D = \bigcap_{i \in I} V_i$  be domain of finite character (i.e.,  $V_i$ 's are valuation overrings of  $D$ , and every  $x \in D^*$  is a non-unit in only a finite number of  $V_i$ 's). Then, if for every  $i \in I$ , except for possibly one of them,  $V_i$  is a DVR, then  $D$  is both IDF and MCD-finite.*

*Proof.* The proof of the [12, Proposition 1], shows that for every  $x \in D^*$ , any set of incomparable divisors of  $x$  is finite, and this fact is sufficient for  $D$  to be both IDF and MCD-finite.  $\square$

Another obvious example of both MCD and MCD-finite domains are GCD domains, but since UFD's are exactly the domains that are both ACCP and GCD, any ACCP domain (respectively, FFD) which is not a UFD, is an example of an MCD (respectively, MCD-finite) domain that is not a GCD domain. In fact, MCD's and GCD's behave quite differently. A rather striking difference between the two concepts was presented by Roitman in [15]. Although it is well known that if any two elements in a domain  $D$  have a GCD then any finite subset of  $D$  also have a GCD, but this is not true for MCD's. In fact, for every  $n \in \mathbb{N}$ , Roitman introduced the class of  $n$ -MCD domains which consists of those domains for which any  $n$  elements have at least one MCD. He then showed that for any  $n \in \mathbb{N}$ , there exists a domain that is  $n$ -MCD but not  $(n+1)$ -MCD ([15, Example 5.2]). In a similar way, we can define  $n$ -MCD-finite domains and we ask the following question:

**Question 1.** *If  $2 \leq n \in \mathbb{N}$ , then does there exist an  $n$ -MCD-finite domain that is not  $(n+1)$ -MCD-finite?*

We recall that a non-zero element  $x$  in a domain  $D$  is called primal, if for every  $a, b \in D^*$ ,  $x \mid ab$  implies that there exist  $x_1, x_2 \in D^*$ , such that  $x = x_1x_2$ ,  $x_1 \mid a$  and  $x_2 \mid b$ . If all the elements of a domain is primal then it is called a pre-Schreier domain and a Schreier domain is an integrally closed, pre-Schreier domain. Also, any GCD domain is Schreier ([6, Theorem 2.4]). We also recall that a domain  $D$  is pre-Schreier if and only if the poset of its principal ideals satisfies Riesz interpolation property, i.e, for every finite subsets  $A$  and  $B$  of principal ideals of  $D$ , if  $A \leq B$  (i.e, for every  $\langle a \rangle \in A$  and  $\langle b \rangle \in B$ ,  $\langle a \rangle \subseteq \langle b \rangle$ ), then there exists a principal ideal  $\langle x \rangle$ , such that  $A \leq \langle x \rangle \leq B$  ([16, Theorem 1.1]). Hence, every finite set of non-zero elements of a pre-Schreier domain, up to associates, has at most one MCD.<sup>1</sup>

**Proposition 3.3.** *Every pre-Schreier domain is MCD-finite. Moreover, a pre-Schreier MCD domain is a GCD domain.*

So in particular, every pre-Schreier domain that is not a GCD domain (see, e.g, the paragraph after [6, Theorem 2.4]), is an example of an MCD-finite domain that is not an MCD domain.

The following proposition is well known.

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<sup>1</sup>The fact that the polynomial extension of a domain that is both IDF and pre-Schreier, is IDF, is also observed by Malcolmson and Okoh in the introduction of [13], and is attributed to Muhammad Zafrullah.

**Proposition 3.4.** *A domain  $D$  is UFD if and only if it is pre-Schreier and ACCP. In particular, any FFD that is not a UFD, is not pre-Schreier (but is both MCD and MCD-finite).*

*Proof.* We first recall that UFD's are exactly GCD domains that are ACCP. Hence, the “only if” part follows. Now let  $D$  be a pre-Schreier ACCP domain. Then by Proposition 3.1,  $D$  is an MCD and hence a GCD domain and so  $D$  is a UFD.  $\square$

Now we give an example of an MCD-finite domain that is neither pre-Schreier nor MCD domain. The domain  $K[Z, \frac{X}{Z^n}, \frac{Y}{Z^n} \mid n \in \mathbb{N} \cup \{0\}]$  in which  $K$  is a field, was used by Roitman as an example of a domain that is not 2-MCD<sup>2</sup> ([15, Example 5.1]). In the next example, we also provide a proof for this fact.

**Example 3.5.** *Let  $D$  be an arbitrary domain. Then the domain*

$$R = D[Z, \{\frac{X}{Z^n}, \frac{Y}{Z^n} \mid n \in \mathbb{N} \cup \{0\}\}]$$

*is not an MCD domain. In fact,  $X$  and  $Y$  do not have any MCD. To see this, let  $f \in \text{CD}_R(X, Y)$ . Note that  $R$  is a subring of*

$$T = (K[Z, \frac{1}{Z}])[X, Y]$$

*and so  $f \in K[Z, \frac{1}{Z}]$ . On the other hand,  $T = (K[X, Y])[Z, \frac{1}{Z}]$  and hence  $f$  is a monomial with respect to the variable  $Z$ . Since for  $n \leq 0$ ,  $Z^n \notin R$ , we have*

$$\text{CD}_R(X, Y) = \{uZ^n \mid u \in U(R), n \in \mathbb{N} \cup \{0\}\}$$

*But, with respect to the relation of divisibility, this set does not have any maximal element, and hence  $\text{MCD}_R(X, Y) = \emptyset$ .*

Now let  $D$  be an FFD which is not a UFD. For example, let  $F \subsetneq K$  be an extension of finite fields and let  $D = F + XK[X]$ . Then, by [1, Proposition 5.2],  $D$  is an FFD. But for any  $c \in K \setminus F$ ,  $(X)(X)$  and  $(cX)(c^{-1}X)$  are both atomic factorizations of  $X^2$  but  $X \not\sim_D cX$  and hence  $D$  is not a UFD. By Proposition 3.4,  $D$  is not pre-Schreier and so neither is  $R$ . But  $R$  is both MCD-finite and IDF. In fact,  $R$  satisfies the following stronger property:

(\*) *The intersection of every infinite set of incomparable principal ideals is 0.*

To see this, suppose on the contrary that  $\{c_i R\}_{i \in I}$  be an infinite set of incomparable principal ideals of  $R$  such that  $\bigcap_{i \in I} c_i R \neq 0$ . By [2, Example 2],  $D[Z, \frac{1}{Z}]$  is an FFD and hence there exist  $i, j \in I$  such that  $i \neq j$  and  $c_i \sim_T c_j$ . But  $c_i \not\sim_R c_j$  and hence there exists an  $n \in \mathbb{N} \cup \{0\}$ , such that  $c_i = Z^n c_j$  which contradicts the incomparability assumption.

**Remark 3.6.** *The condition (\*) in the previous example, is equivalent to the property mentioned in the proof of Theorem 3.2: For every non-zero element  $x$ , any set of incomparable divisor of  $x$ , is finite.*

Also note that the following weaker condition of property (\*), also implies being MCD-finite:

(\*\*) *The intersection of every infinite set of incomparable principal ideals of  $D$  is a principal ideal.*

Indeed, if a domain  $D$  is not MCD-finite, then there exists a non-singleton subset  $A$  of  $D$ , with an infinite number of non-associate MCD's  $\{c_i\}_{i \in I}$ . But these non-associate MCD's are incomparable too, and hence if  $\bigcap_{i \in I} c_i R = cR$  for some  $c \in$

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<sup>2</sup>2-MCD domains are also known as weak GCD.

$D^*$ , then  $c$  would be a common divisor of the elements of  $A$  and since it is a common multiple of  $c_i$ 's, it would be associate to every  $c_i$  which is impossible.

Next, we use  $D + M$  construction to find counterexamples for some of the remaining cases. First, we recite some of the properties of  $D + M$  construction that are relevant to our discussion.

**lemma 3.7.** *Let  $T = K + M$  be a domain, in which  $K$  is a field and  $M$  is a non-zero maximal ideal. Also, let  $D$  be a subfield of  $K$  and  $R = D + M$ . Then:*

- (1) *For every  $m \in M$ ,  $1 + m \in U(R)$  if and only if  $1 + m \in U(T)$ .*
- (2) *For every  $m \in M$ ,  $m \in \mathcal{A}_R$  if and only if  $m \in \mathcal{A}_T$ .*
- (3) *For every  $c_1, c_2 \in K$ ,  $c_1 \sim_R c_2$  if and only if  $c_1 D^* = c_2 D^*$ .*

*Proof.* The proof of parts 1 and 3 are straightforward. For part 2, see [7, Lemma 1.5].  $\square$

**Theorem 3.8.** *Let  $T = K + M$  be a domain, in which  $K$  is a field and  $M$  is a non-zero maximal ideal and let  $D$  be a subfield of  $K$  and  $R = D + M$ . Additionally, suppose that  $M$  contains an atom.<sup>3</sup> Then, if  $R$  is MCD-finite, then the group  $K^*/D^*$  is finite.*

*Proof.* Suppose  $K^*/D^*$  is an infinite set and let  $\{c_i\}_{i \in \alpha}$  be an infinite subset of  $K^*$ , such that  $c_i D^* \neq c_j D^*$  for  $i \neq j$ . Fix two elements  $t$  and  $v$  of  $K^*$ , such that  $t D^* \neq v D^*$ . Let  $m \in M \in \mathcal{A}_R$ . By Lemma 3.7, we conclude that  $\{c_i m\}_{i \in \alpha}$  is an infinite subset of non-associate (irreducible) divisors of  $tm^2$  and  $vm^2$ . On the other hand, all the elements of  $\{\frac{tm^2}{c_i m}\}_{i \in \alpha}$  and  $\{\frac{vm^2}{c_i m}\}_{i \in \alpha}$  are atoms in  $R$  and so  $\{c_i m\}_{i \in \alpha}$  is an infinite set of non-associate maximal common divisors of  $tm^2$  and  $vm^2$ . Therefore,  $R$  is not an MCD-finite domain.  $\square$

**Remark 3.9.** ([4, Theorem 7]) *For any field extension  $F \subseteq K$ , the group  $K^*/F^*$  is finite if and only if  $K = F$  or  $K$  is finite.*

In the next theorem, we recall the behavior of some of the relevant classes of domains in  $D + M$  construction.

**Theorem 3.10.** *Let  $T = K + M$  be a domain, in which  $K$  is a field and  $M$  is a non-zero maximal ideal and let  $D$  be a subring of  $K$  and  $R = D + M$ . Then:*

- (1) [1, Proposition 4.3] *Let  $T$  be quasi-local<sup>4</sup> and suppose  $D$  is not a field. Then,  $R$  is IDF if and only if  $\mathcal{A}_D$ , up to associates, is finite. (See also [1, Proposition 4.2])*
- (2) [1, Proposition 1.2]  *$R$  is ACCP (respectively, atomic) if and only if  $T$  is ACCP (respectively, atomic) and  $D$  is a field.*
- (3) [5, Theorem 11]  *$R$  is a GCD domain if and only if  $T$  is a GCD domain,  $K = \text{frac}(D)$  and  $T_M$  is a valuation domain.*
- (4) [5, Theorem 4]  *$R$  is Noetherian if and only if  $T$  is Noetherian,  $D$  is a field and  $[K : D]$  is finite.*

**Example 3.11.** *The domain  $\mathbb{Q} + X\mathbb{R}[X]$  is a non-Noetherian ACCP (and hence MCD) domain that is not MCD-finite. In fact, even Noetherian domains are not*

<sup>3</sup>Note that by Lemma 3.7 this is unambiguous.

<sup>4</sup>A ring with a unique maximal ideal is said to be quasi-local (whereas “local” is reserved for the case in which the ring is also Noetherian).

necessarily MCD-finite: Consider  $\mathbb{R} + X\mathbb{C}[X]$ . Also,  $\mathbb{Z} + X\mathbb{Q}[[X]]$  is an example of a GCD (and hence MCD and MCD-finite) domain that is not IDF.

Now we are going to find a way to embed an arbitrary domain  $D$  in a domain  $R$  such that  $R$  is not MCD-finite (Of course we could use Theorem 3.8 for this purpose, but in doing so, the factorization structure of the  $D$  is completely lost in  $R$ , as we would have  $\text{frac}(D) \subseteq R$ ).

**Proposition 3.12.** *Let  $D$  be a domain and  $S$  a torsion free, cancellative monoid that is not MCD-finite and set  $R = D[X; S]$ . Then  $R$  is not MCD-finite, the extension  $D \subseteq R$  is division-preserving and  $\mathcal{A}_D \subseteq \mathcal{A}_R$ . Also, in the special case in which  $S$  is reduced, we additionally have  $U(D) = U(R)$ .*

*Proof.* By [11, Theorem 11.1], if  $S$  is torsion-free and cancellative then any divisor of a monomial of  $R = D[X; S]$ , is itself a monomial. All the parts can be deduced from this (See also [9, Lemma 3.1]).  $\square$

**Example 3.13.** (1) *The additive monoid  $C = \{x \in \mathbb{Q} \mid 1 \leq x\} \cup \{0\}$  is not MCD-finite. In fact, let  $A$  be pairwise incomparable subset of  $C$  (i.e., there do not exist  $x, y \in A$ , such that  $1 \leq |x - y|$ ), where  $2 \leq |A|$ . Then, if there exists a  $y \in A$  such that  $y \leq 2$ , then  $\text{MCD}(A) = \{0\}$ , if all the elements of  $A$  are strictly greater than 2, then  $A$  has an infinite number of MCD's and finally if 2 is the minimum element of  $A$ , then  $\text{MCD}(A) = \{1\}$ .*

(2) *Let  $S$  be a cancellative, torsion free and reduced monoid and suppose there exist  $s, t \in S$ , for which  $\text{MCD}(s, t)$  contains at least two (non-associate) elements  $b$  and  $d$  (An elementary example of such monoid is the additive monoid  $(\mathbb{N} \cup \{0\}) \setminus \{1\}$ . Note that 2 and 3 are both MCD of 5 and 6).*

*Set  $T := \prod_{i \in \mathbb{N}} S$ . Let  $s'$  and  $t'$  be the elements of  $T$ , with all the components equal to  $s$  and  $t$  respectively. Let  $c_i$  be the element of  $T$ , with  $b$  in its  $i$ th place, and  $d$  in all the other places. Then  $\{c_i\}_{i \in \mathbb{N}}$  is a set of non-associate MCD of  $s'$  and  $t'$  and hence,  $T$  is not MCD-finite. Moreover, it is easy to see that  $T$  is reduced, cancellative and torsion free.*

Next, we are going to mention another way to embed a domain  $D$  into a domain  $R$  which is not MCD-finite. This time, for a finite non-singleton subset  $A$  of  $D^\#$  that does not generate  $D$ , we construct the domain  $R$  in such a way that the set  $\text{MCD}_R(A)$ , up to associates, becomes infinite.

**Example 3.14.** *Let  $D$  be a domain and  $c_1, \dots, c_n$  non-associate elements in  $D^\#$  such that  $2 \leq n$  and  $D \neq \langle c_1, \dots, c_n \rangle$ . For every  $i \in \mathbb{N} \cup \{0\}$  define the domain  $R_i$  by induction as follows:*

(1) *Set  $R_0 := D$  and  $I_0 := \langle c_1, \dots, c_n \rangle_{R_0}$ .*

(2) *For  $i \in \mathbb{N}$ , set  $R_i := R_{i-1}[X_i, \{\frac{a}{X_i} \mid a \in I_{i-1}\}]$  and  $I_i := \langle c_1, \dots, c_n \rangle_{R_i}$ .*

*Note that for any  $n \in \mathbb{N}$ ,  $I_n \neq R_n$  and hence by [15, Lemma 2.4] and [15, Lemma 2.10] for every  $i \in \mathbb{N}$*

$$X_1, \dots, X_i \in \text{MCD}_{R_i}(c_1, \dots, c_n)$$

*Now set  $R := \bigcup_{i \in \mathbb{N}} R_i$ . By [15, Lemma 2.3(1)], for every  $i, j \in \mathbb{N} \cup \{0\}$  with  $i \leq j$ , the extension  $R_i \subseteq R_j$  is division-preserving and hence, by [15, Lemma 3.1(3)],*

$$\{X_i\}_{i \in \mathbb{N}} \subseteq \text{MCD}_R(c_1, \dots, c_n)$$

*and hence  $R$  is not MCD-finite.*



## 4. STRONGER VERSION OF THEOREM 1.1

Let  $D$  be a domain and  $S \subseteq D^*$ . Define

$$\mathcal{L}(D; S) = D[\{X_s, \frac{s}{X_s} \mid s \in S\}]$$

As we have already mentioned in the introduction, this extension was used by Roitman in [15] to construct an atomic domain such that its polynomial extension is not atomic. Following Roitman's technique, Rand in [14] proved the following result which we present by adding the construction used in the proof.

**Theorem 4.1** ([14, Theorem 2.7]). *Let  $D$  be a domain and  $S$  a subset of  $\mathcal{A}_S$  that is unit-closed. For every  $n \in \mathbb{N} \cup \{0\}$ , we define a domain  $\mathcal{T}_n(D)$  inductively as follows:*

- (1) Set  $\mathcal{T}_0(D; S) := D$ .
- (2) For every  $n \in \mathbb{N}$ , set  $\mathcal{T}_n(D; S) := \mathcal{L}(\mathcal{T}_{n-1}(D); \mathcal{A}_{\mathcal{T}_{n-1}(D)} \setminus S)$ .

Set  $\mathcal{T}^\infty(D; S) := \bigcup_{i \in \mathbb{N}} \mathcal{T}_i(D; S)$ . Then  $U(\mathcal{T}^\infty(D; S)) = U(D)$  and  $\mathcal{A}_{\mathcal{T}^\infty(D; S)} = S$ .

Now we are ready to prove the stronger version of Theorem 1.1.

**Theorem 4.2.** *Let  $D$  be a domain and  $S$  a unit-closed subset of  $\mathcal{A}_D$ . Then there exists a domain  $R$  containing  $D$ , such that  $R$  is not MCD-finite,  $\mathcal{A}_R = S$  and  $D \subseteq R$  is division-preserving.*

*Proof.* Let  $T$  be a cancellative, torsion-free and reduced monoid which is not MCD-finite (e.g., Example 3.13) and let  $B = D[X, T]$ . Then by Proposition 3.13,  $S \subseteq \mathcal{A}_B$ ,  $S$  is unit-closed in  $B$ ,  $D \subseteq B$  is division-preserving and  $B$  is not MCD-finite.

Now if we set  $R := \mathcal{T}^\infty(B; S)$ , then by Theorem 4.1,  $\mathcal{A}_R = S$ . Also by [15, Lemma 3.2(1)] and by induction, for every  $n \in \mathbb{N}$ , the extension  $B \subseteq \mathcal{T}_n(B; S)$  is division-preserving, and so the extensions  $B \subseteq R$  and hence  $D \subseteq R$  are also division-preserving.

Now let  $V$  be a finite subset of  $B^*$  such that  $\text{MCD}_B(V)$ , up to associates, is infinite. Then by [15, Lemma 3.2(6)], for every  $n \in \mathbb{N}$ , we have  $\text{MCD}_B(V) = \text{MCD}_{\mathcal{T}_n}(V)$ . The family  $\{\mathcal{T}_i(B; S)\}_{i \in \mathbb{N}}$  satisfies the conditions of [15, Lemma 3.1] and so by part 4 of that lemma,  $\text{MCD}_B(V) = \text{MCD}_R(V)$ . But  $U(B) = U(R)$  and hence  $R$  is not MCD-finite.  $\square$

**Remark 4.3.** (1) In Theorems 4.1 and 4.2, the assumption for  $S$  to be unit-closed, is not crucial for our purpose. In fact, if we omit this condition, then the only changes are that respectively we would have  $\mathcal{A}_{\mathcal{T}^\infty(D; S)} = U(D)S = \{us \mid u \in U(D), s \in S\}$  and  $\mathcal{A}_R = U(D)S$ .

(2) In Theorem 4.2, if  $S$ , up to associates, is finite, then  $R$  is an IDF domain (Specially if  $S = \emptyset$ , then  $R$  is an antimatter domain) and hence by Theorem 2.1,  $R$  is a counterexample to the ascent of IDF property.

(3) While the construction used by Malcolmson and Okoh relied on  $D$  being countable, the construction in this paper does not, and hence, as conjectured by them in [13, Problem 1], the countability of the domain  $D$  is superfluous in Theorem 1.1.

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